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LARGE DEFLECTION OF CIRCULAR PLATES WITH AFFINE IMPERFECTIONS

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Abstract—The axisymmetric large deflection behaviour of imperfect, tapered, circular plates is analysed by the method of Dynamic Relaxation. The analysis is restricted to plates with linear thickness tapers and affine imperfections, the latter being sympathetic to the applied uniform lateral pressure. In general, the results of the analysis show that the deflections and stresses do not vary greatly with variations in the taper ratio, but that they reduce significantly as the magnitude of the imperfection increases.

NOTATION

a	plate diameter
A_r, A_1, A_t	plate extensional stiffnesses
D_r, D_1, D_t	plate flexural stiffnesses
e_r, e_t	radial and tangential strains
E	Young's modulus
h, h_r	plate thickness at an arbitrary point/radius
h_0	plate thickness at $r = 0$
k	deflection affinity parameter
k_r, k_t	radial and tangential curvatures
M_r, M_t	radial and tangential stress couples
N_r, N_t	radial and tangential stress resultants
q	lateral pressure
$\bar{q}(= qr_0^4E^{-1}h_0^{-4})$	dimensionless lateral pressure
r	arbitrary radius
$r_0(=\frac{1}{2}a)$	reference radius
u	radial displacement
w_0	initial plate deflection
w_1	additional plate deflection
$\bar{w}_1(= w_1/h_0)$	dimensionless additional plate deflection
α	plate thickness taper ratio
ν	Poisson's ratio
$\sigma^b_{r,t}$	radial and tangential bending stress
$\sigma^m_{r,t}$	radial and tangential membrane stress
$\bar{\sigma}^b_{r,t}(= \sigma^b_{r,t}r_0^2E^{-1}h_0^{-2})$	dimensionless radial and tangential bending stress
$\bar{\sigma}^m_{r,t}(= \sigma^m_{r,t}r_0^2E^{-1}h_0^{-2})$	dimensionless radial and tangential bending stress
$(\ )'$	derivative with respect to r

INTRODUCTION

Circular plates are commonly utilised as primary load carrying elements in structures. Usually, these plates are of constant thickness, but sometimes it is convenient to use plates, which taper in thickness. The design of these plates is generally based on small deflection theory. Whilst this approach is adequate in most circumstances, it is nevertheless conservative. A more rational approach would be to utilise elastic large deflection theory and to account, as well, for initial imperfections, which may be present fortuitously or by design.

Unfortunately, the initial imperfections, which may be present, may take a variety of forms and this serves to complicate any study of their influence on the plate behaviour. However, if the restriction is imposed that the initial imperfection should always remain affine to the additional plate deflection, then a parameter study to assess the influence of the initial imperfection on the plate response in the large deflection regime may be more readily accomplished. In the present context, the type of affinity selected for the initial imperfection is linear (see eqn 3 of the next section). This choice implies that it is of the same form as the additional deflection resulting from the applied loading. Thus as only uniform lateral pressures are considered herein, then the implied initial imperfection is a single half-wave. The linear relationship between the initial and additional deflections implies the further consequence that the initial imperfection is load dependent, a fact which is commonly assumed to be atypical of real imperfections. Nevertheless, this feature is thought not to be too serious[1], and, moreover, the results of the present computations may be expected to reveal the same general trends as those based on real imperfections.

Nylander[1] has shown that provided certain boundary condition similarity conditions are satisfied, then the perfect or flat plate large deflection results may be used to construct the corresponding results for plates with affine imperfections. Amongst the many results presented by Nylander are those for a uniformly loaded, constant thickness, simply supported, circular plate with affine imperfections. However, no results were presented for clamped circular plates and, moreover, Nylander has not shown that the same form of construction can be applied to plates, which taper in thickness. Since the usefulness of this technique relies heavily on the availability of the corresponding flat plate solution, the author has not seen fit to examine whether this technique may be extended to tapered plates, even though some perfect tapered plate results for the large deflection regime are available[2, 3]. Instead, the author believes that a more reliable approach is to solve the large deflection equations of tapered plates with affine imperfections directly by numerical integration. The particular integration procedure selected is known as Dynamic Relaxation (DR)[4].

PLATE GEOMETRY

The current study seeks to examine the influence of two particular geometric parameters on the plate response in the presence of a uniform lateral load. These two parameters are: the thickness taper ratio, α, and the imperfection or deflection affinity parameter, k.

In practice, only linear thickness tapers are likely to arise and hence the plate thickness at an arbitrary radius, r, may be expressed as,

$$h_r = h_0(1 - 2\alpha r a^{-1}) \tag{1}$$

where $r = 0 \rightarrow \frac{1}{2}a$. Thus for the present purposes, it is considered sufficient to restrict the computations to three distinct α values, namely: $-\frac{1}{2}$, 0 and $+\frac{1}{2}$.

The total plate deflection may be expressed as,

$$w = w_0 + w_1 \tag{2}$$

where w_0 and w_1 are the initial and additional deflections respectively. If the initial deflection is to remain affine to the additional deflection, then the following relationship is implied,

$$w_0 = kw_1 \tag{3}$$

and hence by varying the value of k a whole range of initial imperfection magnitudes may be examined. The particular k values chosen for computational purposes are: 0, $\frac{1}{2}$, 1 and 2.

ANALYSIS

(i) *Governing equations*

The DR method is suited to the numerical integration of low order differential equations. Because of this, it is convenient to retain the separate identities of the compatibility, constitutive and equilibrium equations throughout the analysis. These equations assume the following forms for axisymmetric deformations:

(a) *Compatibility equations.* Utilising eqns (2) and (3) the compatibility equations may be expressed as,

$$\begin{aligned} e_r &= u^{\cdot} + \tfrac{1}{2}(1+2k)w_1^{\cdot 2} \\ e_t &= r^{-1}u \\ k_r &= -w_1^{\cdot\cdot} \\ k_t &= -r^{-1}w_1^{\cdot}. \end{aligned} \tag{4}$$

(b) *Constitutive equations.*

$$
\begin{aligned}
N_r &= A_r e_r + A_1 e_t \\
N_t &= A_1 e_r + A_t e_t \\
M_r &= D_r k_r + D_1 k_t \\
M_t &= D_1 k_r + D_t k_t.
\end{aligned}
\tag{5}
$$

Equations (5) apply to polar orthotropic materials, but all the plates considered herein are assumed to be fabricated from an isotropic material (e.g. steel, for which $\nu = 0.3$) and hence,

$$A_r = A_t = Eh(1-\nu^2)^{-1}, \quad A_1 = \nu A_r$$

and (6)

$$D_r = D_t = Eh^3(1-\nu^2)^{-1}/12, \quad D_1 = \nu D_r.$$

(c) *Equilibrium equations.* Only the transverse equilibrium equation is affected by the affine imperfection, so that these equations are

$$
\begin{aligned}
&N_r' + r^{-1}(N_r - N_t) = 0 \\
&M_r'' + r^{-1}(2M_r' - M_t') + (1+k)\{N_r(w_1'' + r^{-1}w_1') + N_r' w_1'\} + q = 0.
\end{aligned}
\tag{7}
$$

(ii) *Boundary conditions*

The study is restricted to circular plates with either a simply supported or a clamped boundary and in each case full in-plane restraint is assumed. Thus the appropriate boundary condition equations are:

(1) *Simply supported* ($r = \frac{1}{2}a$).

$$
\begin{aligned}
w_1 &= 0 \\
M_r &= D_r k_r + D_1 k_t = 0 \\
u &= 0.
\end{aligned}
\tag{8a}
$$

(2) *Clamped* ($r = \frac{1}{2}a$).

$$
\begin{aligned}
w_1 &= 0 \\
w_1' &= 0 \\
u &= 0.
\end{aligned}
\tag{8b}
$$

(3) DR *procedure.*

As the DR method is fully documented elsewhere [4], a description of its application to eqns (4)–(8) is omitted. Nevertheless, it is worth pointing out that the present application does make use of rationally determined fictitious densities [5] together with a unit time increment. Thus only the two damping factors require to be determined by arbitrary means and their evaluation and adjustment to achieve rapid solution convergence is a relatively simple matter. This approach has been found to be superior to the more usual approach, in which both the fictitious densities and the damping factors are determined by trial and error.

RESULTS

A preliminary set of computations was undertaken in order to: (a) verify the computer program, and (b) to determine a suitable mesh size for the main computations. Table 1, which compares the DR results with those of Nylander for a uniformly loaded, constant thickness, simply supported, circular plate, clearly demonstrates objective (a). Based on these and other results, it was concluded that objective (b) would be met by using a 10.5 interval interlacing mesh for the main computations of the parameter study.

In order that the computer results may enjoy their widest possible application, they are presented in Figs. 1–5 in nondimensional graphical form, which allows interpolation with reasonable accuracy.

Table 1. Comparison between Nylander's and the DR program results at the centre of a uniformly loaded, constant thickness, simply supported, circular plate with an affine imperfection ($\nu = 0.25$)

Pressure $\bar{q}$	Affinity k	Nylander's Results* $\bar{w}_1$	Nylander's Results* $\bar{\sigma}^m_{r,t}$	DR Program Results $\bar{w}_1$	DR Program Results $\bar{\sigma}^m_{r,t}$
20	0.36	1.40	2.90	1.3684	2.9331
20	1.00	1.00	2.75	1.0020	2.7648
20	4.44	0.45	2.25	0.4828	2.1456

* The Nylander Results have been evaluated by scaling graphs and are therefore only given to two decimal places.

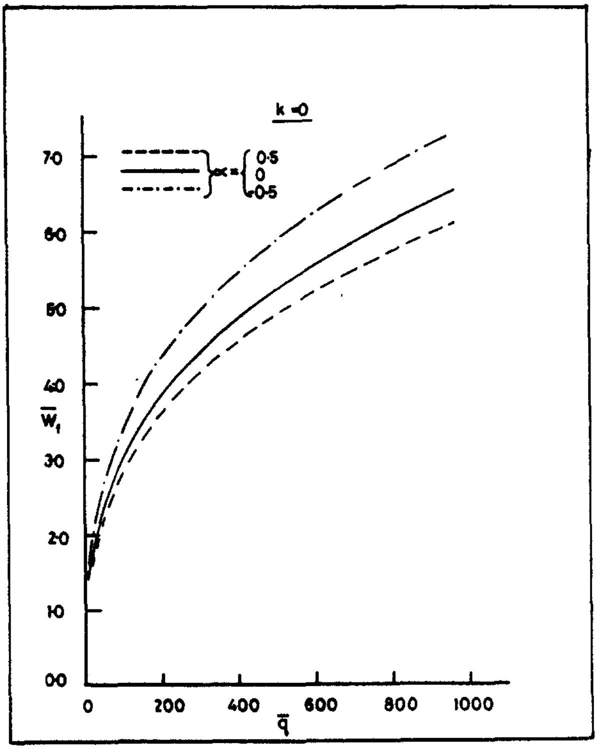

Fig. 1(a).

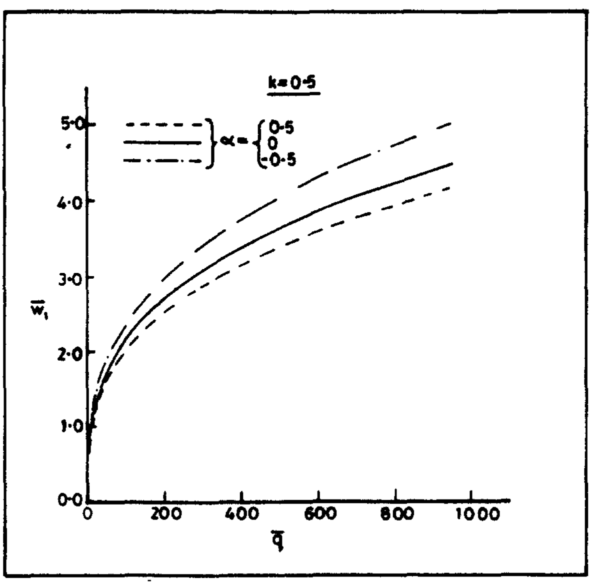

Fig. 1(b).

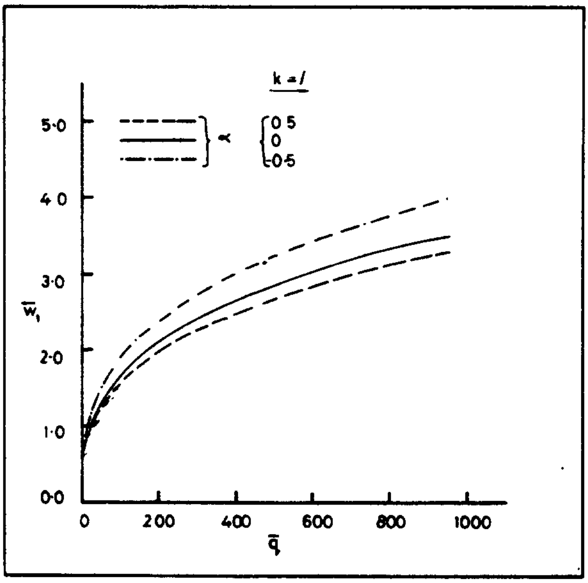

Fig. 1c).

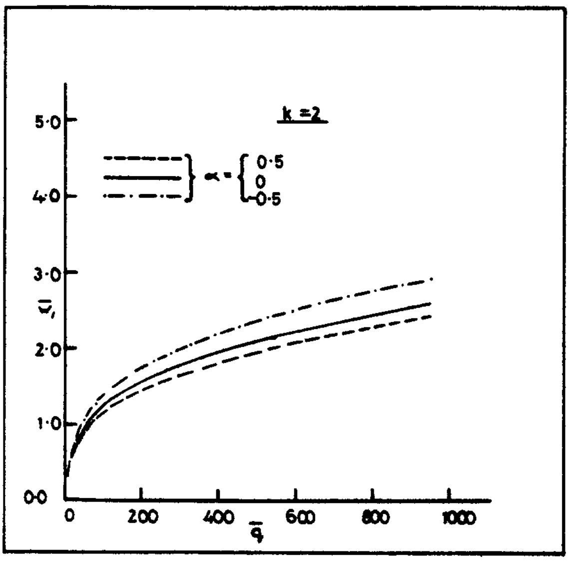

Fig. 1(d).

Fig. 1. Additional central deflection vs lateral pressure for a simply supported circular plate ($\nu = 0.3$); (a) $k = 0$, (b) $k = 0.5$, (c) $k = 1$, (d) $k = 2$.

Figure 1 presents the additional deflection at the centre of a simply supported plate as a function of the lateral pressure for three values of the taper ratio and four values of the affinity parameter. It is evident that, in all cases, the additional deflection increases marginally with increasing taper ratio and decreases substantially as the value of the affinity parameter increases. Corresponding results for the clamped plate are not presented, since they differ only marginally from the simply supported results, i.e. from about 5% at low values of the lateral pressure to rather less than 2% at higher values. Therefore, except where great accuracy is required, the results of Fig. 1 may be used for both sets of boundary conditions.

The central bending stress vs lateral pressure is plotted in Fig. 2 for a simply supported plate. Here too, the dependence of the bending stress on the value of the taper ratio and the affinity parameter is similar to that of Fig. 1, i.e. marginal in the case of the former and rather more substantial in the case of the latter.

Figure 3 compares the simply supported and clamped plate centre bending stress for $\alpha = \pm 0.5$ and $k = 1$ and 2. It is evident that the stresses diverge as the taper ratio decreases, but that this effect reduces as k increases in value.

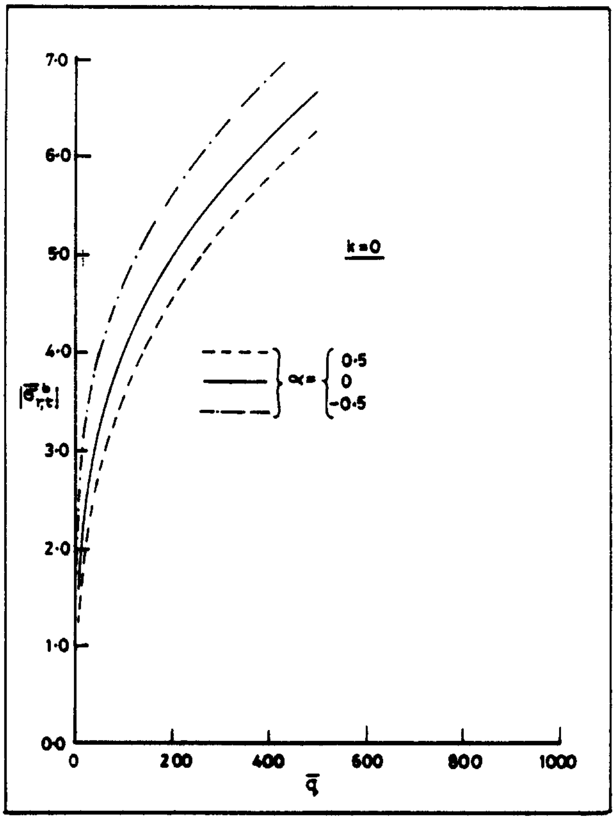
7·0
6·0
5·0
4·0
3·0
2·0
1·0
0·0
0
200
400
600
800
1000
q̄
k = 0
α = 0·5
0
−0·5

Fig. 2(a).

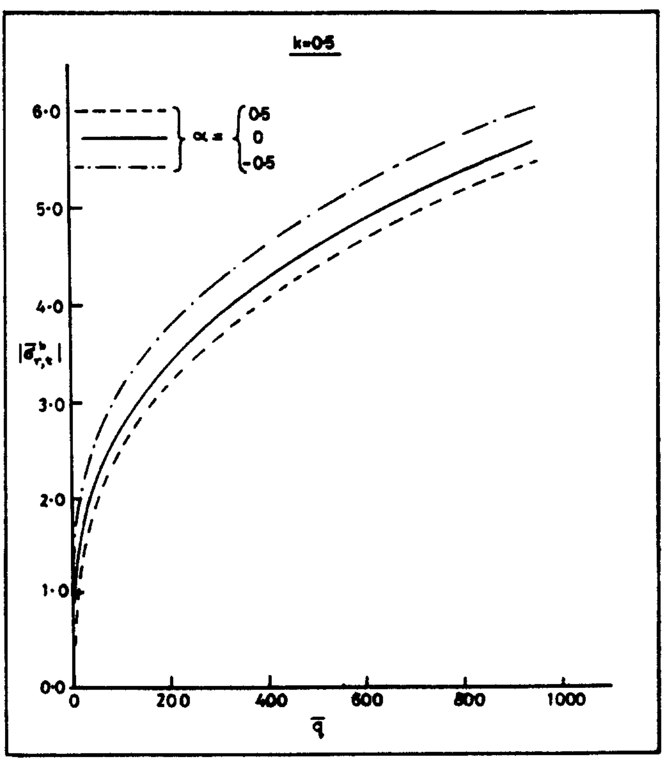
k=0·5
6·0
5·0
4·0
3·0
2·0
1·0
0·0
0
200
400
600
800
1000
q̄
α = 0·5
0
−0·5

Fig. 2(b).

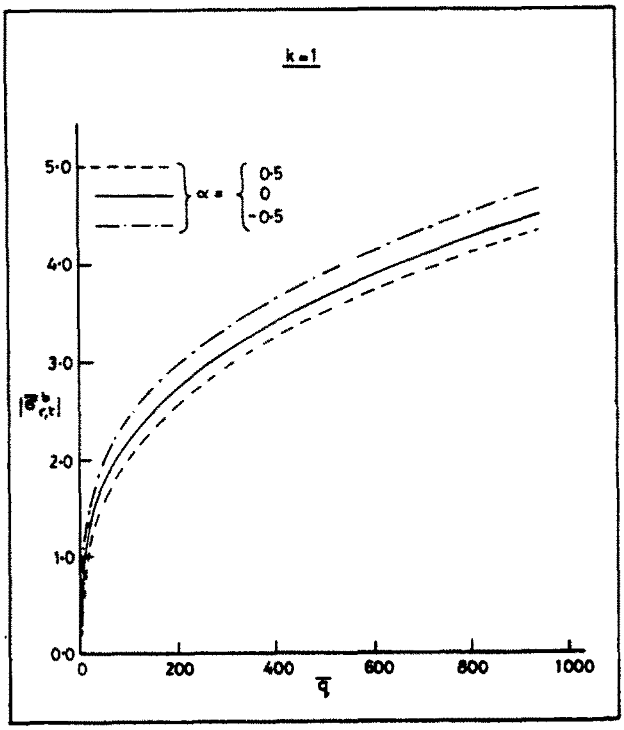

Fig. 2(c).

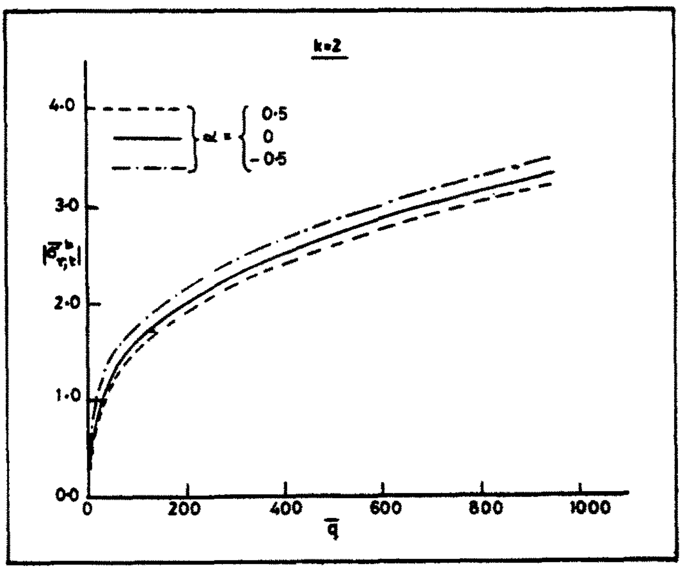

Fig. 2(d).

Fig. 2. Central bending stress vs lateral pressure for a simply supported circular plate ($\nu = 0.3$); (a) $k = 0$, (b) $k = 0.5$, (c) $k = 1$, (d) $k = 2$.

In Fig. 4 the simply supported plate centre membrane stress vs lateral pressure curves are plotted for the upper and lower taper ratio limits and for each of the four k values considered herein. They demonstrate that this stress is almost independent of the plate taper ratio and, moreover, that it reduces slightly as the value of the affinity parameter, k, increases.

Finally, Fig. 5 compares the plate centre membrane stress for both simply supported and clamped boundary conditions. Although comparative results are shown for the upper limit of the taper ratio only, a similar behaviour is observed for other α and k values. Thus, these curves suggest that the results presented in Fig. 4 may be used for plates with either boundary condition without an undue loss in accuracy.

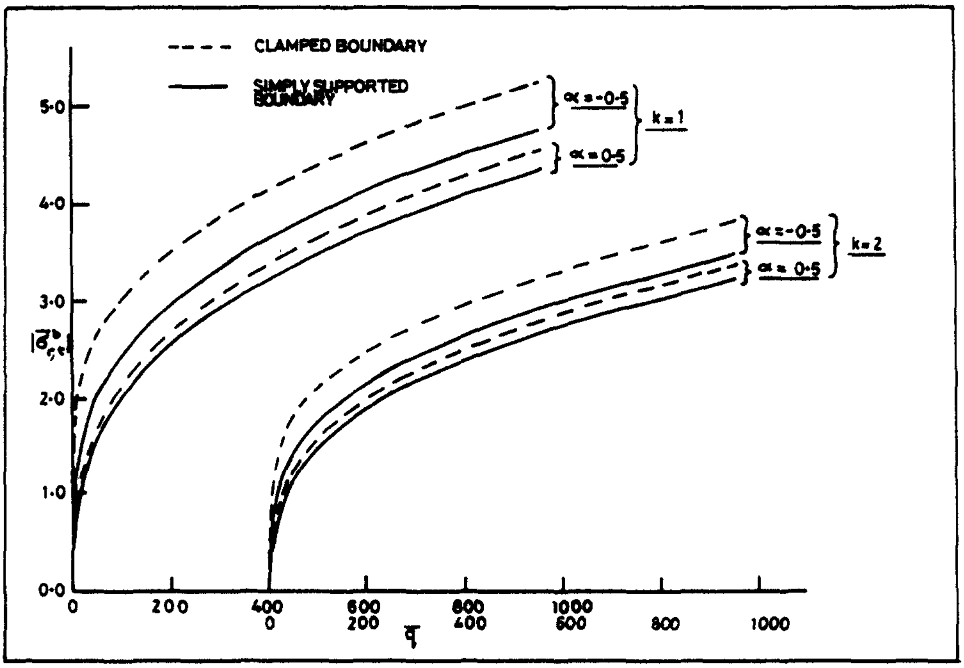

Fig. 3. Central bending stress vs lateral pressure for simply supported and clamped circular plates ($\nu = 0.3$).

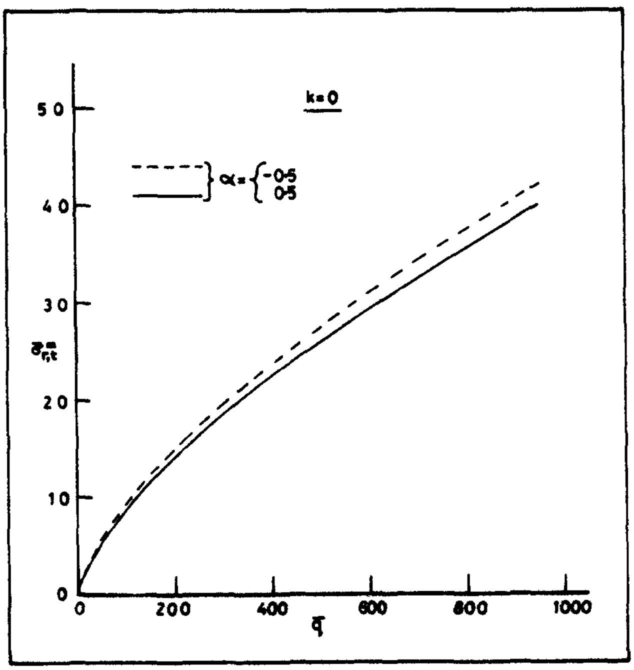

Fig. 4(a).

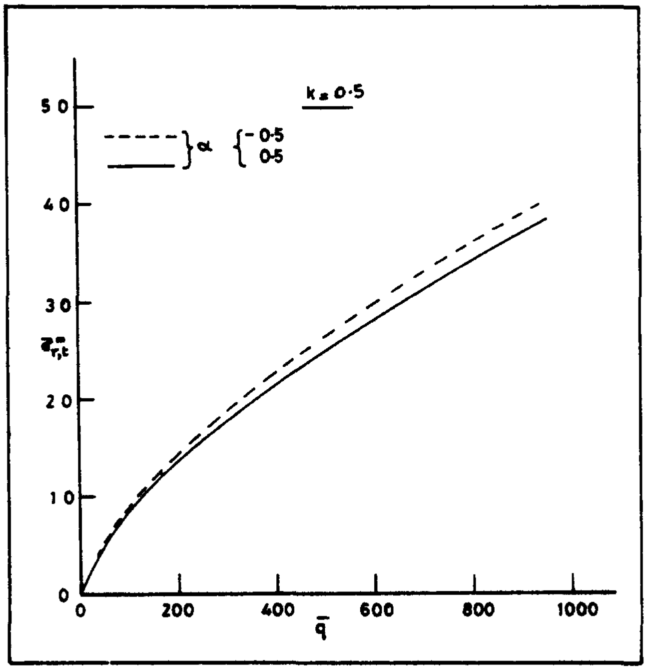
k = 0·5
α { -0·5
0·5
50
40
30
20
10
0
$\bar{\sigma}^{m}_{r,t}$
0
200
400
600
800
1000
$\bar{q}$

Fig. 4(b).

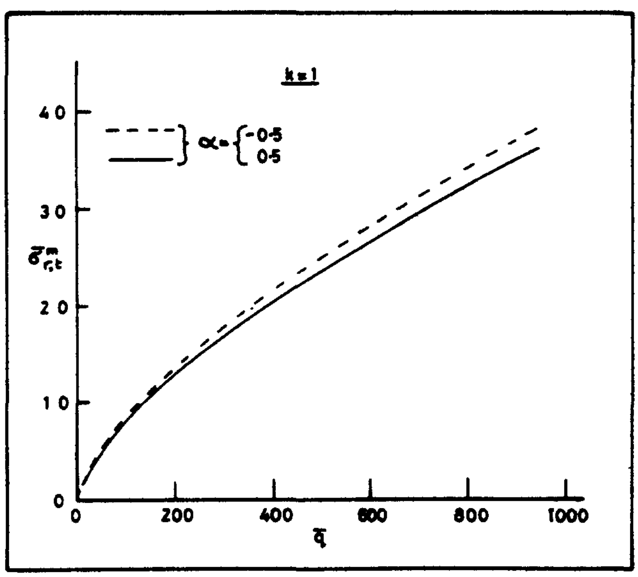
k = 1
α = { -0·5
0·5
40
30
20
10
0
$\bar{\sigma}^{m}_{r,t}$
0
200
400
600
800
1000
$\bar{q}$

Fig. 4(c).

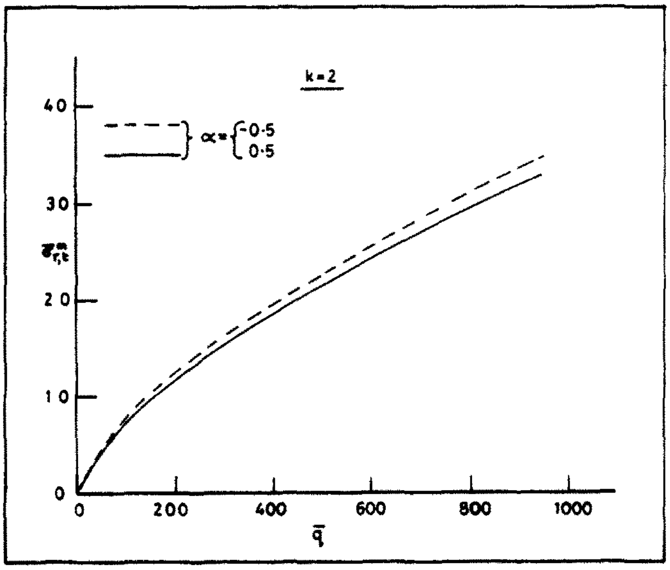

Fig. 4(d).

Fig. 4. Central membrane stress vs lateral pressure for a simply supported circular plate ($\nu = 0.3$); (a) $k = 0$, (b) $k = 0.5$, (c) $k = 1$, (d) $k = 2$.

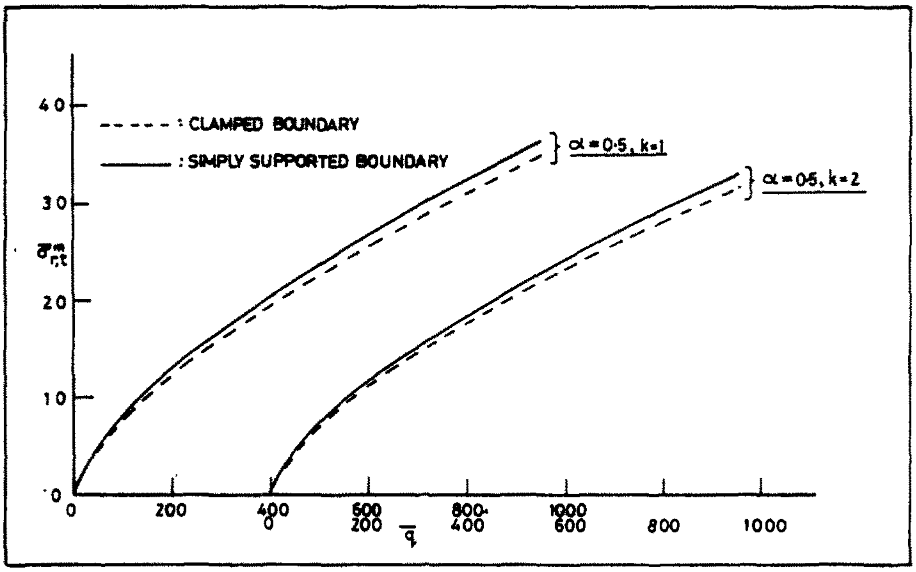

Fig. 5. Central membrane stress vs lateral pressure for simply supported and clamped circular plates ($\nu = 0.3$).

CONCLUSIONS

The main conclusions to be drawn from the computed results are summarised as follows:

(1) The central deflection increases moderately with increasing taper ratio and reduces substantially as the affinity parameter increases.

(2) The central bending stress decreases moderately with increasing taper ratio and reduces substantially as the affinity parameter increases.

(3) The central membrane stress is relatively insensitive to taper ratio and decreases only marginally as the affinity parameter increases.

(4) The central deflection and membrane stress are relatively insensitive to the boundary support conditions.

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